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Abstract: In a recent paper authors gave an analysis of the exact controllability problem via the moment problem approach. Namely, the steering conditions of controllable states are formulated as a vectorial moment problem using some Riesz basis. One of the main difficulties was the choice of the basis as, in general, a basis of eigenvectors does not exist. In this contribution we use a change of control by a feedback law and modify the structure of the system in such a way that there exists a basis of eigenvectors which allows a simpler expression of the moment problem. Hence, one obtains the result on exact controllability and on the time of exact controllability.

Keywords: neutral-type systems, exact controllability, moment problem, Riesz basis, distributed delays.

1. INTRODUCTION

There exist many approaches to study controllability problems for delay systems (see e.g. Morse (1976); Manitius and Triggiani (1978); O'Connor and Tarn (1983) and references therein). One of the most powerful approaches is to consider a delay system as a system in some functional space:

$$\dot{x} = \mathscr{A}x + \mathscr{B}u,$$

where $x \in H$, H being a Hilbert space, $\mathscr{A}$ is the generator of a C_0-semigroup.

In contrast to finite-dimensional systems, Kalman controllability concept ($\mathscr{R}_T = H$, $\mathscr{R}_T$ is the reachability set from 0 at time T), generally, does not hold for infinite-dimensional systems, in particular for somes classes of systems with delays. However, for neutral type systems it is possible to pose the problem of reaching the set $\mathscr{D}(\mathscr{A})$, what leads to the notion of the exact controllability in this sense.

Consider a neutral type system given by

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + Lz_t + Bu, \quad t \geq 0, \tag{1}$$

where A_{-1}, B are constant matrices of dimensions $n \times n$ and $n \times r$, respectively; $z_t : [-1, 0] \to \mathbb{C}^n$ is the history of z defined by $z_t(s) = z(t+s)$; the delay operator L is given by

$$Lf = \int_{-1}^{0} A_2(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}f(\theta)\ \mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)f(\theta)\ \mathrm{d}\theta,$$

where A_2, A_3 are $n \times n$-matrices whose elements belong to $L_2([-1, 0], \mathbb{C})$. For this system, the following criterium of controllability had been recently obtained by coauthors of the present paper (Rabah and Sklyar (2007a)).

Theorem 1. The system (1) is exactly null-controllable if and only if the following conditions are verified.

(i) There is no $\lambda \in \mathbb{C}$ and $y \in \mathbb{C}^n$, $y \neq 0$, such that $\Delta^*_{\mathscr{A}}(\lambda)y = 0$ and $B^*y = 0$, where

$$\begin{aligned}\Delta_{\mathscr{A}}(\lambda) &= \lambda I - \lambda \mathrm{e}^{-\lambda} A_{-1} \\ &\quad -\lambda\int_{-1}^{0} \mathrm{e}^{\lambda s} A_2(s)\mathrm{d}s - \int_{-1}^{0} \mathrm{e}^{\lambda s} A_3(s)\mathrm{d}s,\end{aligned} \tag{2}$$

or equivalently $\operatorname{rank}(\Delta_{\mathscr{A}}(\lambda)\ B) = n$ for all $\lambda \in \mathbb{C}$.

(ii) There is no $\mu \in \sigma(A_{-1})$ and $y \in \mathbb{C}^n, y \neq 0$, such that $A^*_{-1}y = \bar{\mu}y$ and $B^*y = 0$, or equivalently

$$\operatorname{rank}(B \quad A_{-1}B \cdots A_{-1}^{n-1}B) = n.$$

If conditions (i) and (ii) hold, then the system is controllable at the time $T > n_1$ and not controllable at the time $T \leq n_1$, where n_1 is the controllability index of the pair (A_{-1}, B).

An important contributions of this result consist in giving the precise time of controllability. This may be very important in the problem of time minimal problem and other related problems of optimal control. In this case the semigroup is not explicitly known in contrast to the situation of several discrete delays (see O'Connor and Tarn (1983); Jacobs and Langenhop (1976); Banks et al. (1975); Rivera Rodas and Langenhop (1978)). To study controllability we use the moment problem approach. Namely, the steering conditions of controllable states are interpreted as a vectorial moment problem with respect to a special Riesz basis. We analyze the solvability of the obtained non-Fourier trigonometric moment problem using methods developed by Avdonin and Ivanov (1995).

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When there exists a basis of the state space consisting of eigenvectors, the expression of the moment problem is simplified (see Rabah and Sklyar (2007b)). For the system (1), the existence of a basis of generalized eigenvectors is determined by the form of the matrix A_{-1}. It is now well known that, in general, such a basis does not exist (see Rabah et al. (2003, 2005)). Due to the last, the procedure of the choice of a Riesz basis is quite sophisticated in a general case. Besides, the complicated form of the obtained basis makes technically difficult further manipulations with it.

Later we observed that by means of a change of control it is possible to pass over to an equivalent controllability problem for a system with a matrix A_{-1} of a simple structure. For such system there exists a Riesz basis of eigenvectors and the form of the corresponding moment problem is much simpler. This makes the proofs of the main results more illustrative what motivated us for writing this paper. Here we give the proof of the Theorem 1 for the system (1) with A_{-1} of a special form and show that this fact implies the proof for a system with an arbitrary matrix A_{-1}.

More detailed surveys on the problem of exact controllability of neutral type systems may be found in Rabah and Sklyar (2007a) and references therein.

We consider the operator model of the neutral-type system (1) introduced by Burns et al. (1983) in product space. The state space is $M_2(-1,0;\mathbb{C}^n) = \mathbb{C}^n \times L_2(-1,0;\mathbb{C}^n)$, shortly M_2, and (1) can be reformulated as

$$\dot{x}(t) = \mathscr{A}x(t) + \mathscr{B}u(t), \quad \mathscr{A} = \begin{pmatrix} 0 & L \\ 0 & \dfrac{\mathrm{d}}{\mathrm{d}\theta} \end{pmatrix}, \mathscr{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}, \tag{3}$$

with $\mathscr{D}(\mathscr{A}) = \{(y, z(\cdot)) \in M_2 : z \in H^1([-1,0];\mathbb{C}^n), y = z(0) - A_{-1}z(-1)\}$.
If $L = 0$, i.e. if $A_2(\theta) = A_2(\theta) \equiv 0$, the operator $\mathscr{A}$ is noted by $\widetilde{\mathscr{A}}$.

The reachability set from 0 at time T is defined by

$$\mathscr{R}_T = \left\{ x : x = \int_0^T \mathrm{e}^{\mathscr{A}t}\mathscr{B}u(t)\,\mathrm{d}t, \quad u(\cdot) \in L_2(0,T;U) \right\}.$$

It can be shown that $\mathscr{R}_T \subset \mathscr{D}(\mathscr{A})$ for all $T > 0$ (see Ito and Tarn (1985)). We say that the system (3) is exactly null-controllable by controls from L_2 at the time T if $\mathscr{R}_T = \mathscr{D}(\mathscr{A})$. This means that the set of solutions of the system (1), $\{z(t), t \in [T-1, T]\}$, coincides with $H^1([T-1,T];\mathbb{C}^n)$. We denote by $X_{\mathscr{A}}$ the space $\mathscr{D}(\mathscr{A}) \subset M_2$ with the graph norm.

The paper is organized as follows. In Section 2 we simplify our system using a change of control and in Section 3 we expand the steering condition using a spectral Riesz basis. Section 4 is devoted to solvability of a moment problem. In Sections 5-6 we give a short proof of our main result. Finally we give an illustrative example and concluding remarks.

2. MODIFICATION OF THE SYSTEM

We begin with the following proposition.

Lemma 2. The system (1) is exactly null-controllable at time T if and only if the perturbed system

$$\dot{z}(t) = (A_{-1} + BP)\dot{z}(t-1) + Lz_t + Bu \tag{4}$$

is exactly null-controllable at time T for any $P \in \mathbb{C}^{n\times r}$.

Proof. Obviously it is enough to prove one implication only. Assume that the system (1) is controllable at the time T. This means that for any function $f(t) \in H^1(T-1,T;\mathbb{C}^n)$ there exists a control $u(t) \in L_2(0,T;\mathbb{C}^n)$ such that the solution of the equation

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + Lz_t + Bu(t), \tag{5}$$

with the initial condition $z(t) = 0$, $t \in [-1,0]$, verifies $z(t) = f(t)$, $t \in [T-1,T]$. Let us rewrite (5) in the form

$$\dot{z}(t) = (A_{-1} + BP)\dot{z}(t-1) + Lz_t + Bv(t),$$

where $v(t) = u(t) - P\dot{z}(t-1)$, $t \in [0,T]$. Since $z(t-1) \in H^1(0,T;\mathbb{C}^n)$, then $v(t) \in L_2(0,T;\mathbb{C}^n)$. Thus, the control $v(t)$ transfers the state $z(t) = 0$, $t \in [-1,0]$ to the state $z(t) = f(t)$, $t \in [T-1,T]$ by virtue of (4). This means that (4) is also controllable at the time T. □

Let us observe that in the conditions of controllability (i) and (ii) of Theorem 1 the matrix A_{-1} may be substituted by the matrix $A_{-1} + BP$ for any P. Indeed, let us denote by $\widehat{\mathscr{A}}$ the operator corresponding to the system (4). Then, the relations $\Delta^*_{\mathscr{A}}(\lambda)y = 0$ and $B^*y = 0$ are equivalent to the relations $\Delta^*_{\widehat{\mathscr{A}}}(\lambda) = [\Delta^*_{\mathscr{A}}(\lambda) - \lambda\mathrm{e}^{-\lambda}P^*B^*]y = 0$ and $B^*y = 0$ with the same y and λ. The equivalency of the conditions $\operatorname{rank}(B \quad A_{-1}B \ \cdots \ A_{-1}^{n-1}B) = n$ and $\operatorname{rank}(B \quad (A_{-1} + BP)B \ \cdots \ (A_{-1} + BP)^{n-1}B) = n$ is a well-know classical result (see e.g. Wonham (1985)).

Thus, we conclude that if we prove Theorem 1 for a system (1) with a matrix $\widehat{A}_{-1}$ of the form $\widehat{A}_{-1} = A_{-1} + BP$, we also prove this theorem for the system with the matrix A_{-1}.

Let us fix n distinct real numbers:

$$\{\mu_1, \ldots, \mu_n\} \subset \mathbb{R}, \quad \mu_i \neq \mu_j, i \neq j, \qquad \mu_i \notin \{0,1\}. \tag{6}$$

There exists a matrix $P \in \mathbb{C}^{r\times n}$ such that $\sigma(A_{-1} + BP) = \{\mu_m\}_{m=1}^n$ (see e.g. Wonham (1985)).

Thus, without loss of generality, we may assume that A_{-1} is such that

$$\sigma(A_{-1}) = \{\mu_m\}_{m=1}^n. \tag{7}$$

Due to the construction $\det A_{-1} \neq 0$. We denote by $\{c_m\}_{m=1}^n$ the basis of normed eigenvectors of A_{-1}.

3. RIESZ BASIS OF EIGENVECTORS AND EXPANSION OF THE STEERING CONDITION

3.1 The Riesz basis

Let us recall that we denote by $\widetilde{\mathscr{A}}$ the operator $\mathscr{A}$ in the case $L = 0$, i.e. when $A_2 = A_3 = 0$.

It easy to see that the eigenvalues of the operator $\widetilde{\mathscr{A}}$ are of the form

$$\sigma(\widetilde{\mathscr{A}}) = \{\widetilde{\lambda}_m^k = \ln|\mu_m| + 2k\pi\mathrm{i},\ m = 1, \ldots, n,\ k \in \mathbb{Z}\} \cup \{0\},$$

where $\{\mu_1, \ldots, \mu_n\} = \sigma(A_{-1})$. Due to the specific structure of A_{-1}, the operator $\widetilde{\mathscr{A}}$ possesses simple eigenvalues only: no root-vectors and to each $\widetilde{\lambda}_m^k$ corresponds only one eigenvector $\widetilde{\varphi}_{m,k}$. We norm these eigenvectors such that

$$\inf_{k\in\mathbb{Z}} \|\widetilde{\varphi}_{m,k}\| = \widetilde{\rho} > 0, \quad \sup_{k\in\mathbb{Z}} \|\widetilde{\varphi}_{m,k}\| = \widetilde{R} < \infty.$$

The spectrum of $\mathscr{A}$ allows the following characterization

$$\sigma(\mathscr{A}) = \{\ln|\mu_m| + 2k\pi\mathrm{i} + \overline{\mathrm{o}}(1/k),\ m = 1, \ldots, n,\ k \in \mathbb{Z}\}.$$

There exists $N \in \mathbb{N}$ such that the total multiplicity of the eigenvalues of $\mathscr{A}$, contained in the circles $L_m^k(r^{(k)})$, equals 1 for all $m = 1,\dots,n$ and $k : |k| > N$, where $L_m^k(r^{(k)}) = L_m^k$ are circles centered at $\widetilde{\lambda}_m^k$ and their radii $r^{(k)}$ satisfy the relation $\sum_{k\in\mathbb{Z}} (r^{(k)})^2 < \infty$ (Rabah et al., 2008, Theorem 4). We denote these eigenvalues of the operator $\mathscr{A}$ as λ_m^k and the corresponding eigenvectors as $\varphi_{m,k}$, $m = 1,\dots,n$, $|k| > N$. Now the vectors $\varphi_{m,k}$ are normed such that $P_m^{(k)}\widetilde{\varphi}_{m,k} = \varphi_{m,k}$, where $P_m^{(k)} = \frac{1}{2\pi \mathrm{i}}\int_{L_m^{(k)}} R(\lambda,\mathscr{A})\mathrm{d}\lambda$. The families $\{\varphi_{m,k}\}$ and $\{\widetilde{\varphi}_{m,k}\}$ are quadratically close: $\sum_{|k|>N}\sum_{m=1}^{n} \|\varphi_{m,k} - \widetilde{\varphi}_{m,k}\|^2 < \infty$, what, in particular, implies

$$\inf_{|k|>N} \|\varphi_{m,k}\| = \rho > 0, \qquad \sup_{|k|>N} \|\varphi_{m,k}\| = R < \infty. \tag{8}$$

Outside the circles $L_m^{(k)}$, $|k| > N$, $m = 1,\dots,n$, there is only a finite number of eigenvalues of $\mathscr{A}$ noted $\widehat{\lambda}_s$, $s = 1,\dots,N$, and counted with multiplicities. We denote by $\widehat{\varphi}_s$, $s = 1,\dots,N$, the corresponding generalized eigenvectors of the operator $\mathscr{A}$. The family

$$\{\varphi\} = \{\varphi_{m,k}\} \cup \{\widehat{\varphi}_s\} \tag{9}$$

forms a Riesz basis of M_2 (Rabah et al. (2005)).

Let us denote by

$$\{\psi\} = \{\psi_{m,k}\} \cup \{\widehat{\psi}_s\} \tag{10}$$

the family of eigenvectors of the adjoint operator $\mathscr{A}^*$ which is biorthogonal to $\{\varphi\}$. Here $\mathscr{A}^*\psi_{m,k} = \overline{\lambda_m^k}\psi_{m,k}$, $m = 1,\dots,n$, $|k| > N$ and $s = 1,\dots,N$. This family also forms a Riesz basis of M_2.

3.2 The steering condition

Let us expand the steering condition $x_T = \big(y_T,\, z_T(\cdot)\big) = \int_0^T \mathrm{e}^{\mathscr{A}t}\mathscr{B}u(t)\,\mathrm{d}t$ with respect to the basis $\{\varphi\}$ and to the biorthogonal basis $\{\psi\}$. A state $x = (y,\, z(\cdot)) \in M_2$ is reachable at time T if and only if

$$\sum_{\varphi\in\{\varphi\}} \langle x,\psi\rangle\,\varphi = \sum_{\varphi\in\{\varphi\}} \int_0^T \left\langle \mathrm{e}^{\mathscr{A}t}\mathscr{B}u(t),\psi\right\rangle \mathrm{d}t\cdot\varphi.$$

Let $\{b_1,\dots,b_r\}$ be an arbitrary basis of the image of the matrix B, and $\mathbf{b}_d = (b_d,\, 0)^T \in M_2$, $d = 1,\dots,r$. Then the steering condition can be substituted by the following system of equalities:

$$\begin{aligned}\langle x,\psi\rangle &= \int_0^T \left\langle \mathrm{e}^{\mathscr{A}t}\mathscr{B}u(t),\psi\right\rangle \mathrm{d}t\\ &= \sum_{d=1}^{r}\int_0^T \left\langle \mathrm{e}^{\mathscr{A}t}\mathbf{b}_d,\psi\right\rangle u_d(t)\,\mathrm{d}t,\end{aligned} \tag{11}$$

where $\psi \in \{\psi\}$, $u(\cdot) \in L_2(0,T;\mathbb{C}^r)$. Let us write the term $\langle \mathrm{e}^{\mathscr{A}t}\mathbf{b}_d,\psi\rangle$ for $\psi = \psi_{m,k}$, $m = 1,\dots,n$, $|k| > N$ as follows:

$$\begin{aligned}\left\langle \mathrm{e}^{\mathscr{A}t}\mathbf{b}_d,\, \psi_{m,k}\right\rangle_{M_2} &= \mathrm{e}^{\lambda_m^k t}\langle \mathbf{b}_d,\psi_{m,k}\rangle_{M_2}\\ &= \mathrm{e}^{\lambda_m^k t}\langle b_d, y_{m,k}\rangle_{\mathbb{C}^n},\end{aligned} \tag{12}$$

where $y_{m,k} \in \operatorname{Ker}\Delta^*_{\mathscr{A}}(\overline{\lambda_m^k})$. We introduce the notation:

$$q_{m,k}^d = k\langle \mathbf{b}_d,\psi_{m,k}\rangle_{M_2}. \tag{13}$$

Due to (12), the infinite part of the system (11), corresponding to $\psi \in \{\psi_{m,k},\ |k| > N, m = 1,\dots,n\}$, reads as

$$k\left\langle \begin{pmatrix} y_T\\ z_T(\cdot)\end{pmatrix},\psi_{m,k}\right\rangle = \sum_{d=1}^{r}\int_0^T \mathrm{e}^{\lambda_m^k t}\, q_{m,k}^d u_d(t)\,\mathrm{d}t. \tag{14}$$

Next we observe that if $\psi = \widehat{\psi}_s$, $s = 1,\dots,N$, then

$$\left\langle \mathrm{e}^{\mathscr{A}t}\mathbf{b}_d,\psi\right\rangle, = \left\langle \mathbf{b}_d, \mathrm{e}^{\mathscr{A}^*t}\psi\right\rangle = \widehat{q}_s^d(t)\,\mathrm{e}^{\widehat{\lambda}_s t},$$

where $\widehat{q}_s^d(t)$ is some polynomial. Therefore, the finite part of (11), corresponding to $\psi \in \{\widehat{\psi}_s\}$, reads as

$$\left\langle \begin{pmatrix} y_T\\ z_T(\cdot)\end{pmatrix},\widehat{\psi}_s\right\rangle = \sum_{d=1}^{r}\int_0^T \mathrm{e}^{\widehat{\lambda}_s t}\,\widehat{q}_s^d(t)u_d(t)\,\mathrm{d}t. \tag{15}$$

Thus, we observe that the state $\big(y_T,\, z_T(\cdot)\big) \in M_2$ is reachable from 0 at the time $T > 0$ if and only if the equalities (14) and (15) hold for some controls $u_d(\cdot) \in L_2(0,T)$, $d = 1,\dots,r$. The obtained moment problem is the main object of our further analysis.

4. THE PROBLEM OF MOMENTS AND THE RIESZ BASIS PROPERTY

Consider a collection of functions $\{g_k(t),\ t \in [0,\infty[\}_{k\in\mathbb{N}}$ assuming that for any $k \in \mathbb{N}$, $T > 0$: $g_k(\cdot) \in L_2(0,T)$, and consider the following problem of moments:

$$s_k = \int_0^T g_k(t)u(t)\,\mathrm{d}t, \qquad k \in \mathbb{N}. \tag{16}$$

The following well-known fact is a consequence of Bari theorem (see Gohberg and Krein (1969) and Young (1980)).

Proposition 3. The following statements are equivalent:

(i) For the scalars s_k, $k \in \mathbb{N}$, the problem (16) has a solution $u(\cdot) \in L_2(0,T)$ if and only if $\{s_k\} \in \ell_2$, i.e., $\sum_{k\in\mathbb{N}} s_k^2 < \infty$;

(ii) the family $\{g_k(t)\}_{k\in\mathbb{N}}$, $t \in [0,T]$ forms a Riesz basis in the closure of its linear span

$$\operatorname{Cl}\operatorname{Lin}\{g_k(t),\ k \in \mathbb{N}\}.$$

Let us put $\mathscr{L}(0,T) \overset{\text{def}}{=} \operatorname{Cl}\operatorname{Lin}\{g_k(t),\ k \in \mathbb{N}\} \subset L_2(0,T)$. The following propositions from (Rabah and Sklyar (2007a)) are used later.

Proposition 4. Let us suppose that for some $T_1 > 0$ the functions $\{g_k(t)\}_{k\in\mathbb{N}}$, $t \in [0,T_1]$, form a Riesz basis in $\mathscr{L}(0,T_1) \subset L_2(0,T_1)$ and $\operatorname{codim}\mathscr{L}(0,T_1) < \infty$. Then for any $0 < T < T_1$, there exists an infinite-dimensional subspace $\ell^T \subset \ell_2$ such that the problem of moments (16) is unsolvable for $\{s_k\} \in \ell^T$ if $\{s_k\} \neq \{0\}$.

Proposition 5. Let us consider the moment problem

$$s_k = \sum_{d=1}^{r}\int_0^T g_k^d(t)u_d(t)\,\mathrm{d}t, \qquad k \in \mathbb{N}, \tag{17}$$

where $\sum_{k\in\mathbb{N}} \int_0^T |g_k^d(t)|^2\,\mathrm{d}t < \infty$, $d = 1,\dots,r$. Then the set $S_{0,T}$ of sequences $\{s_k\}$ for which problem (17) is solvable is a nontrivial submanifold of ℓ_2, i.e., $S_{0,T} \neq \ell_2$.

Let $\delta_1,\dots,\delta_n$ be different, modulus $2\pi\mathrm{i}$, complex numbers, let N be natural integer and let the set $\{\varepsilon_{m,k},\ |k| > N, m =$

$1,\dots,n\} \subset \mathbb{C}^n$ be such that $\sum_{m,k} |\varepsilon_{m,k}|^2 < \infty$. Let $\widetilde{\mathscr{E}}_N$ be the family

$$\widetilde{\mathscr{E}}_N = \left\{ \mathrm{e}^{(\delta_m + 2\pi \mathrm{i} k + \varepsilon_{m,k})t}, \quad |k| > N, m = 1,\dots,n \right\}.$$

Next, let $\varepsilon_1,\dots,\varepsilon_r$ be another collection of different complex numbers such that $\varepsilon_j \neq \delta_m + 2\pi \mathrm{i} k + \varepsilon_{m,k}$, $j = 1,\dots,r$, $m = 1,\dots,n$, $|k| > N$, and let $m'_1,\dots,m'_r$ be positive integers. Let us denote by $\mathscr{E}_0$ the collection

$$\mathscr{E}_0 = \left\{ \mathrm{e}^{\varepsilon_j t}, t\,\mathrm{e}^{\varepsilon_j t}, \dots, t^{m'_s - 1}\,\mathrm{e}^{\varepsilon_j t},\ j = 1,\dots,r \right\}.$$

The following theorem, which is based on results of Avdonin and Ivanov (1995), is the main tool of our further analysis.

Theorem 6. (i) If $\sum_{j=1}^{r} m'_j = (2N+1)n$, then the family

$$\mathscr{E} = \widetilde{\mathscr{E}}_N \cup \mathscr{E}_0$$

constitutes a Riesz basis in $L_2(0,n)$.

(ii) If $T > n$, then independently of the number of elements in $\mathscr{E}_0$, the family $\mathscr{E}$ forms a Riesz basis of the closure of its linear span in the space $L_2(0,T)$.

Now we apply Theorem 6 to the collection of functions appearing in (14). Let us fix $d \in \{1,\dots,r\}$ and choose an arbitrary subset $L \subset \{1,\dots,n\}$.

Theorem 7. For any choice of d, L, for any $T \geq n' = |L|$ the collection of functions

$$\Phi_1 = \left\{ \mathrm{e}^{\lambda_m^k t}\, q_{m,k}^d, \quad |k| > N;\ m \in L \right\}$$

constitutes a Riesz basis of $\mathrm{Cl\,Lin}\Phi_1$ in $L_2(0,T)$.

If $T = n'$, the subspace $\mathrm{Cl\,Lin}\Phi_1$ is of finite codimension $(2N+1)n'$ in $L_2(0,n')$.

Proof. Consider the linear operator $\mathscr{T} : \mathrm{Lin}\,\Phi_1 \to \mathrm{Lin}\,\Phi_1$ defined on the elements of Φ_1 by the equalities

$$\mathscr{T}(\mathrm{e}^{\lambda_m^k t}\, q_{m,k}^d) = \mathrm{e}^{\lambda_m^k t}$$

for $|k| > N$, $m \in L$. It can be proved that the family $\{q_{m,k}^d\}$ is uniformly bounded, and then it follows from Theorem 6 that the operator $\mathscr{T}$ is bounded in the sense of $L_2(0,T)$ and its extension to $\mathrm{Cl\,Lin}\,\Phi_1$ is a bounded one-to-one operator from L to L. Hence, since the images of the elements of Φ_1 form a Riesz basis of $\mathrm{Cl\,Lin}\,\Phi_1$ (Theorem 6), then Φ_1 is also a Riesz basis of this subspace of $L_2(0,T)$.

Finally, let us observe that in the case $T = n'$ the space $\mathrm{Cl\,Lin}\,\Phi_1$ is of codimension $(2N+1)n'$ in $L_2(0,T)$ (see Theorem 6). Then Φ_1^c which is an orthonormal complement of the basis of Φ_1 to a basis of $L_2(0,T)$, consists of exactly $(2N+1)n'$ elements. $\square$

5. THE NECESSARY CONDITION OF CONTROLLABILITY

Let us study the solvability of the systems of equalities (14) and (15). Consider the sequence of functions

$$\left\{ \int_0^T \mathrm{e}^{\lambda_m^k t}\, q_{m,k}^d u(t)\,\mathrm{d}t, \quad |k| > N \right\} \tag{18}$$

for any fixed d and $u(\cdot) \in L_2(0,T)$. It follows from Theorem 7 that all nonzero functions of the collection $\left\{ \mathrm{e}^{\lambda_m^k t}\, q_{m,k}^d \right\}_{|k|>N}$ form a Riesz basis of their linear span in $L_2(0,T')$ if T' is large enough. Therefore, by Proposition 3, the sequence (18) belongs to the class ℓ_2. This gives the following proposition.

Proposition 8. If the state $\left(y_T,\ z_T(\cdot)\right)$ is reachable from 0 by (3), then it satisfies the following equivalent conditions:

(C1) $\displaystyle\sum_{|k|>N} \sum_{m=1}^{n} k^2 \left| \left\langle \begin{pmatrix} y_T \\ z_T(\cdot) \end{pmatrix}, \psi_{m,k} \right\rangle \right|^2 < \infty,$

(C2) $\displaystyle\sum_{|k|>N} \sum_{m=1}^{n} k^2 \left\| P_m^{(k)} \begin{pmatrix} y_T \\ z_T(\cdot) \end{pmatrix} \right\|^2 < \infty,$

(C3) $\begin{pmatrix} y_T \\ z_T(\cdot) \end{pmatrix} \in \mathscr{D}(\mathscr{A}).$

From Proposition 8 it follows (see also Ito and Tarn (1985)), that the set $\mathscr{R}_T$ of the states reachable from 0 by virtue of the system (3) and controls from $L_2(0,T)$ is always a subset of $\mathscr{D}(\mathscr{A})$.

Theorem 9. Assume that the system (3) is null-controllable by controls from $L_2(0,T)$ for some $T > 0$. Then the following two conditions hold:

(i) There is no $\lambda \in \mathbb{C}$ and $y \in \mathbb{C}^n, y \neq 0$, such that $\Delta_{\mathscr{A}}^*(\lambda) y = 0$ and $B^* y = 0$, or equivalently $\mathrm{rank}\,(\Delta_{\mathscr{A}}(\lambda) \quad B) = n$ for all $\lambda \in \mathbb{C}$.

(ii) There is no $\mu \in \sigma(A_{-1})$ and $y \in \mathbb{C}^n, y \neq 0$, such that $A_{-1}^* y = \bar{\mu} y$ and $B^* y = 0$, or equivalently

$$\mathrm{rank}\,(B \quad A_{-1}B \cdots A_{-1}^{n-1}B) = n.$$

Proof. The condition (i) may be reformulated as follows: there is no eigenvector g of the adjoint operator $\mathscr{A}^*$ belonging to $\mathrm{Ker}\mathscr{B}^*$. Let (i) be false. Then there exists a vector $g \neq 0$ such that $\mathscr{A}^* g = \lambda g$ and $g \in \mathrm{Ker}\mathscr{B}^*$. Consider an arbitrary state $\left(y_T,\ z_T(\cdot)\right) \in \mathscr{R}_T$. This gives

$$\left\langle \begin{pmatrix} y_T \\ z_T(\cdot) \end{pmatrix}, g \right\rangle = \int_0^T \left\langle u(t), \mathscr{B}^*\,\mathrm{e}^{\mathscr{A}^* t} g \right\rangle \mathrm{d}t = 0.$$

This means that $\mathscr{R}_T$ is not dense in M_2 and so cannot be equal to $\mathscr{D}(\mathscr{A})$ which is dense in M_2 ($\mathscr{A}$ is an infinitesimal generator). Hence null-controllability is impossible.

Now let us suppose that condition (ii) does not hold, i.e., there exists $y \neq 0$ such that $A_{-1}^* y = \bar{\mu}_m y$ and $B^* y = 0$. Due to the special form of A_{-1} we have $y = c_m$. Let us fix the index m and consider the subset of equalities (14):

$$s_k = k \left\langle \begin{pmatrix} y_T \\ z_T(\cdot) \end{pmatrix}, \psi_{m,k} \right\rangle = \sum_{d=1}^{r} \int_0^T \mathrm{e}^{\lambda_m^k t}\, q_{m,k}^d u_d(t)\,\mathrm{d}t, \tag{19}$$

where $|k| > N$, $q_{m,k}^d = k \langle \mathbf{b}_d, \psi_{m,k} \rangle_{M_2}$. From Proposition 5 and some technical estimates from the steering condition, it follows that the set of solvability of (19) is a linear submanifold $\ell' \subset \ell_2$, $\ell' \neq \ell_2$. We conclude that there exist sequences $\{s_k\}_{|k|>N}$ for which (19) is not solvable. This means that there exist states $\left(\, y_T,\ z_T(\cdot) \,\right)$ that satisfy (C1) but are not reachable from 0 by the system (3). Thus $\mathscr{R}_T \neq \mathscr{D}(\mathscr{A})$. $\square$

6. THE SUFFICIENT CONDITION OF CONTROLLABILITY

First we consider single control systems. We need the following preliminarie result.

Lemma 10. For the system (3) let there exist a natural N and $T_0 > 0$ such that the moment problem (14) for $T = T_0$ and $|k| > N$ is solvable for all sequences satisfying (C1). From condition (i) of Theorem 9 it follows $\mathscr{R}_T = \mathscr{D}(\mathscr{A})$ as $T > T_0$.

Proof. The condition of the Lemma implies that R_{T_0} is of finite co-dimension and then in order to obtain $\mathrm{R}_T = \mathscr{D}(\mathscr{A})$ we use a corollary of Hahn-Banach Theorem. □

Theorem 11. Let the system (3) is of single control ($r = 1$) and let conditions (i) and (ii) of Theorem 9 hold. Then

(1) the system (3) is null-controllable at the time $T > n$;
(2) the estimation of the time of controllability in (1) is exact, i.e. the system is not controllable if $T = n$.

If the delay is equal to h the time controllability is $T = nh$.

Proof. Let us observe that all the eigenspaces of $\mathscr{A}^*$ and $\widetilde{\mathscr{A}^*}$ are one-dimensional. Indeed, otherwise there exists an eigenvector g of $\mathscr{A}^*$ (or $\widetilde{\mathscr{A}^*}$) such that $\langle \mathbf{b}, g\rangle_{M_2} = 0$. Eigenvectors of the adjoint operator have the form $g = (y, z(\theta))^T$, where y is nonzero and satisfies $\Delta^*_{\mathscr{A}}(\lambda_0)y = 0$ (or $\Delta^*_{\widetilde{\mathscr{A}}}(\lambda_0)y = 0$) for some λ_0. Since $\langle \mathbf{b}, g\rangle_{M_2} = 0$ gives $\langle b, y\rangle_{\mathbb{C}^n} = 0$ we arrive at a contradiction with the conditions of Theorem 9.

Thus, equalities (14) and (15) take, in our case, the form

$$k\left\langle \begin{pmatrix} y_T \\ z_T(\cdot) \end{pmatrix}, \psi_{m,k} \right\rangle = \int_0^T \mathrm{e}^{\lambda_m^k t} q_{m,k} u(t)\,\mathrm{d}t, \tag{20}$$

where $|k| > N$, $m = 1, \dots, n$, and

$$\left\langle \begin{pmatrix} y_T \\ z_T(\cdot) \end{pmatrix}, \widehat{\psi}_s \right\rangle = \int_0^T \mathrm{e}^{\widehat{\lambda}_s t} \widehat{q}_s(t) u(t)\,\mathrm{d}t, \tag{21}$$

where $s = 1, \dots, N$. From the condition (i) it follows that all $q_{m,k} \neq 0$ and all polynomials $\{\widehat{q}_s(t)\}$ are nontrivial. Let us introduce the following notation

$$\begin{aligned} \Phi_1 &= \left\{ \mathrm{e}^{\lambda_m^k t} q_{m,k},\ |k| > N,\ m = 1, \dots, n \right\}, \\ \widehat{\Phi} &= \left\{ \mathrm{e}^{\widehat{\lambda}_s t} \widehat{q}_s(t),\ s = 1, \dots, N \right\}. \end{aligned}$$

For a large enough N, the collection $\Phi = \Phi_1 \bigcup \widehat{\Phi}$ forms a Riesz basis in $\mathrm{Cl\,Lin}\,\Phi \subset L_2(0, T)$ (Theorem 7). Then by Proposition 3 the moment problem (20) is solvable if and only if (C1) holds. Due to Lemma 10, this yields $\mathscr{R}_T = \mathscr{D}(\mathscr{A})$ for $T > n$.

To prove the assertion (2) we first recall that the total number of elements of the family $\widehat{\Phi}$ equals to $\sum_{m=1}^{\ell} \widehat{p}_{m,1} = (2N+2)n$. On the other hand, it follows from Theorem 7 that in $L_2(0, n)$ we have

$$\operatorname{codim} \mathrm{Cl\,Lin}\,\Phi_1 = (2N+1)n.$$

This means that the family $\Phi = \Phi_1 \cup \widehat{\Phi}$ contains at least n functions, which are presented as linear combinations of the others. As a consequence, the set of reachability $\mathscr{R}_T$ for $T = n$ cannot be equal to $\mathscr{D}(\mathscr{A})$. More precisely, the codimension of $\mathscr{R}_T$ in $\mathscr{D}(\mathscr{A})$ satisfies the estimation $n \leq \operatorname{codim} \mathscr{R}_T < \infty$. □

We note that the system (3) is also uncontrollable at time $T < n$. Moreover, it follows from Proposition 4 that, in this case, the set $\mathrm{Cl}\,\mathscr{R}_T$ is of infinite codimension in $X_{\mathscr{A}}$.

Let us now consider the multivariable case: $\dim B = r > 1$. Let $\{b_1, \dots, b_r\}$ be an arbitrary basis noted β. Denote $B_i = (b_{i+1}, \dots, b_r)$, $i = 0, 1, \dots, r-1$, which gives in particular $B_0 = B$ and $B_{r-1} = (b_r)$ and we put formally $B_r = 0$. We introduce the integers

$$\begin{aligned} m_i^\beta \ = \ & \operatorname{rank}(B_{i-1} \quad A_{-1}B_{i-1} \cdots A_{-1}^{n-1}B_{i-1}) \\ & - \operatorname{rank}(B_i \quad A_{-1}B_i \cdots A_{-1}^{n-1}B_i) \end{aligned} \tag{22}$$

corresponding to the basis β. Let us denote

$$m_1 = \max_\beta m_1^\beta, \qquad \overline{m} = \min_\beta \max_i m_i^\beta,$$

for all choices of a basis β. It is easy to show that for all β, there exists i such that $m_i^\beta \geq m_1$ and then $\overline{m} \geq m_1$.

Now we can formulate the main result of this section.

Theorem 12. Let conditions (i) and (ii) of Theorem 9 hold. Then

(1) the system (3) is null-controllable at the time $T > \overline{m}$;
(2) the system (3) is not controllable at the time $T < m_1$.

If the delay is h instead of 1, then in (1) and (2) $\overline{m}$ and m_1 must be replaced by $\overline{m}h$ and $m_1 h$, respectively.

Proof. The proof consists on the analysis of several single moment problems step by step as in Rabah and Sklyar (2007a), where the basis is more complicated. □

To complete our analysis, we obtain the precise time of controllability. Recall that the first index n_1 is the minimal integer ν such that $\operatorname{rank}(B,\ A_{-1}B,\ \dots,\ A_{-1}^{\nu-1}B) = n$, if the pair (A_{-1}, B) is controllable. One can easily show that $m_1 \leq n_1 \leq \overline{m}$. It is well known that in contrast to indices $m_1, \overline{m}$, the controllability index n_1 is invariant under feedback. This means that n_1 is the same for all couples $(A_{-1} + BP, B)$. Then one can choose a feedback matrix P and a basis in $\mathbb{C}^n$ such that $A_{-1} + BP$ take the form (see Wonham (1985))

$$F = \operatorname{diag}\{F_1, \dots, F_r\},\ F_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1^i & a_2^i & a_3^i & \cdots & a_{n_i}^i \end{pmatrix}$$

and B becomes $G = \operatorname{diag}\{g_1, \dots, g_r\}$, where $g_i = (0, 0, \dots, 1)^{\mathrm{T}}$, the dimension being $n_i \times 1$. It is easy to check that $\overline{m}(F, G) = m_1(F, G) = n_1$. Moreover, the spectrum of F may be chosen arbitrarily by means of an appropriate choice of P.

The following result concludes our considerations.

Theorem 13. Let the neutral-type system (1) be in the general form. Conditions (i) and (ii) of Theorem 9 are necessary and sufficient for the exact controllability of the system. Under these conditions, the precise time of controllability is $T = n_1$. This means that the system is not controllable for $T \leq n_1$ and is controllable for $T > n_1$. If the delay is h instead of 1, then the exact time of controllability is $n_1 h$.

Proof. The necessity and sufficiency of the condition (i) and (ii) are proved by Theorem 9 and Theorem 12. We need only to precise the time of controllability.

If the conditions (i) and (ii) are verified for (1), then they are also verified for any perturbed system. From (ii) we choose a matrix P such that $A_{-1} + BP$ is nonsingular and $\overline{m}(A_{-1} + BP, B) = m_1(A_{-1} + BP, B) = n_1$. Thus, the perturbed system is exactly null-controllable at the time $T > n_1$ and is not controllable at $T < n_1$. By Lemma 2 we infer that our system (1) satisfies the same condition.

Moreover, it is easy to prove, arguing as in the proof of Theorem 11, that the system (1) is also not controllable at the time $T = n_1$. More precisely, the codimension of $\mathscr{R}_{n_1}$ in $X_{\mathscr{A}}$ is finite and not less than n_1. For $T < n_1$, the codimension of $\mathscr{R}_T$ is infinite. □

7. EXAMPLE

Consider a three-dimensional system given by the equation (1) with

$$A_{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -11 & 16 \\ 0 & -9 & 13 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},$$

and the matrices $A_s(\theta) = \{a^s_{ij}(\theta)\}^3_{i,j=1}$, $s = 2,3$ are of lower-triangular form ($a^s_{ij}(\theta) \equiv 0$ for $i < j$).

The spectrum $\sigma(A_{-1}) = \{1\}$ and the corresponding Jordan block is three-dimensional. We simplify the system by means of the feedback change $u(t) = v(t) + P\dot{z}(t-1)$ with

$$P = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 9 & -16 \end{pmatrix}.$$

Thus, we obtain

$$\widehat{A}_{-1} = A_{-1} + BP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}. \tag{23}$$

Let $\mathscr{A}$ be the operator corresponding to the modified system. By construction, the characteristic matrix $\Delta_{\mathscr{A}}(\lambda)$ defined by (2) is of lower-triangular form. Thus, the eigenvalues λ^k_m of $\mathscr{A}$ are the roots of the following three equations:

$$\lambda + m\lambda\,\mathrm{e}^{-\lambda} - \int_{-1}^{0} \mathrm{e}^{\lambda s}\left[\lambda a^2_{mm}(s) + a^3_{mm}(s)\right] \mathrm{d}s = 0, \tag{24}$$

where $m = 1,2,3$. Let us rewrite the steering condition (14) and first we calculate the values of $q^d_{m,k}$. The eigenvector of $\mathscr{A}^*$ corresponding to the eigenvalue $\overline{\lambda^k_m}$ is of the form $\psi_{m,k} = (y_{m,k},\ C_{m,k}(\theta)y_{m,k})^T$, where $y_{m,k} \in \mathrm{Ker}\Delta^*_{\mathscr{A}}(\overline{\lambda^k_m})$ (Rabah et al. (2008)). To satisfy the estimation (8) and taking into account that $\Delta^*_{\mathscr{A}}(\lambda)$ is of upper-triangular form, we chose $y_{m,k} \in \mathrm{Ker}\Delta^*_{\mathscr{A}}(\overline{\lambda^k_m})$ as follows:

$$y_{1,k} = \frac{1}{k}(1,0,0), \quad y_{2,k} = \frac{1}{k}(1,\alpha_k,0), \quad y_{3,k} = \frac{1}{k}(1,\beta_k,\gamma_k),$$

The sequences $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\gamma_k\}$ are bounded. Calculating $q^d_{m,k}$ given by (13), we obtain

$$\begin{array}{lll} q^1_{1,k} = 1, & q^1_{2,k} = 1, & q^1_{3,k} = 1, \\ q^2_{1,k} = 0, & q^2_{2,k} = \alpha_k, & q^2_{3,k} = \beta_k + \gamma_k. \end{array}$$

Thus, the moment problem (14) takes the following form:

$$\begin{aligned} k\langle x_T,\ \psi_{1,k}\rangle &= \int_0^T \mathrm{e}^{\lambda^k_1 t}\, u_1(t)\,\mathrm{d}t, \\ k\langle x_T,\ \psi_{2,k}\rangle &= \int_0^T \mathrm{e}^{\lambda^k_2 t}\left[u_1(t) + \alpha_k u_2(t)\right]\mathrm{d}t, \\ k\langle x_T,\ \psi_{3,k}\rangle &= \int_0^T \mathrm{e}^{\lambda^k_3 t}\left[u_1(t) + (\beta_k + \gamma_k)u_2(t)\right]\mathrm{d}t. \end{aligned}$$

The pair $(A_{-1},\ B)$ is controllable and its controllability index n_1 equals to 2. Moreover, $B^* y_{m,k} \neq 0$ for all $m = 1,2,3$. Thus, conditions (i) and (ii) are satisfied and, applying Theorem 13, we obtain that the system is exactly controllable and the time of controllability is $T = 2$.

8. CONCLUSION

We give a new approach to the problem of the exact controllability by the moment problem method. The difficulty of the choice of basis is contounred by a change of control using state feedback. This change of control allowed us to simplify the structure of the system, what makes the proof of the criterium of exact controllability more illustrative. This approach may be used for more general neutral term given by the difference operator $Kf = \sum_{i=1}^{r} A_{h_i} f(h_i)$, $h_i \in [-1,0]$. This idea offers some new perspective for the analysis of controllability and also stabilizability of general neutral type systems.

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